

Multi-time Lagrangian 1-forms for families of Bäcklund transformations. Relativistic Toda-type systems

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Abstract

We establish the pluri-Lagrangian structure for families of Bäcklund transformations of relativistic Toda-type systems. The key idea is a novel embedding of these discrete-time (one-dimensional) systems into certain two-dimensional pluri-Lagrangian lattice systems. This embedding allows us to identify the corner equations (which are the main building blocks of the multi-time Euler-Lagrange equations) with local superposition formulae for Bäcklund transformations. These superposition formulae, in turn, are key ingredients necessary to understand and to prove commutativity of the multi-valued Bäcklund transformations. Furthermore, we discover a two-dimensional generalization of the spectrality property known for families of Bäcklund transformations. This result produces a family of local conservation laws for two-dimensional pluri-Lagrangian lattice systems, with densities being derivatives of the discrete 2-form with respect to the Bäcklund (spectral) parameter. Thus, a relation of the pluri-Lagrangian structure with more traditional integrability notions is established.

1 Introduction

This paper can be considered as a continuation of our recent paper [BPS13], where we gave an application of the general Lagrangian theory of discrete integrable systems of classical mechanics, developed in [Sur13], to families of Bäcklund transformations for non-relativistic Toda-type systems. The development of the general theory in [Sur13] was prompted by an example of the discrete time Calogero-Moser system studied in [YKLN11], and belongs to the line of research on Lagrangian theory of discrete integrable systems initiated by Lobb and Nijhoff in [LN09] and followed by a number publications [LNQ09, LN10, BS10a, ALN12, BS12, BS14, BPS14a, BPS14b]. The notion of integrability of discrete systems, lying at the basis of this development, is that of the multidimensional consistency. This understanding of integrability has been a major breakthrough [BS02, Nij02], and stimulated an impressive activity boost in the area, cf. [BS08].

The original idea of Lobb and Nijhoff can be summarized as follows: solutions of integrable systems deliver critical points simultaneously for actions along all manifolds of the corresponding dimension in multi-time; the Lagrangian form is closed on solutions. This idea resembles the

classical notion of pluriharmonic functions and, more generally, of pluriharmonic maps [Rud69, OV90, BFPP93], which are simultaneous extremals of the Dirichlet energy along all holomorphic curves in a multi-dimensional complex vector space. This motivated us in [BPS14a, BS14] to introduce a novel term for the new branch of the calculus of variations specific for integrable systems: we call the corresponding systems *pluri-Lagrangian*, and we argue that integrability of variational systems should be understood as the existence of the pluri-Lagrangian structure. In the present paper, we hope to provide an additional evidence in favor of this view.

Here, we establish and investigate the pluri-Lagrangian structure for a more general class of systems than the one studied in [BPS13], namely for the so-called relativistic Toda-type systems. The general form of a relativistic Toda-type system is

$$\ddot{x}_k = r(\dot{x}_k)(f(x_{k+1} - x_k) - f(x_k - x_{k-1}) + \dot{x}_{k+1}g(x_{k+1} - x_k) - \dot{x}_{k-1}g(x_k - x_{k-1})).$$

The general form of a discrete-time relativistic Toda-type system is

$$G(\tilde{x}_k - x_k) - G(x_k - \underline{x}_k) = H(x_{k+1} - x_k) - H(x_k - x_{k-1}) + F(\underline{x}_{k+1} - x_k) - F(x_k - \tilde{x}_{k-1}).$$

A theory and an exhaustive list of integrable systems of this type can be found in [AS97a, AS97b, Adl99, Sur03, BS10b]. To derive a pluri-Lagrangian structure for these (one-dimensional) systems, it turned out to be necessary to re-interpret them as a particular case of two-dimensional lattice systems, and to develop a general theory of discrete two-dimensional pluri-Lagrangian systems. The latter goal was achieved in [BPS14a].

The structure and the main results of the present paper are as follows.

- In section 2, we provide the reader with an overview of the theory of pluri-Lagrangian systems in dimensions $d = 1, 2$, following mainly [Sur13, BPS14a]. The fundamental notion of consistent systems of 2D, resp. 3D corner equations, which are main building blocks of pluri-Lagrangian systems, is reminded in detail.
- Then, in section 3, we present the construction of two mutually commuting families of symplectic maps (Bäcklund transformations) from a generic discrete two-dimensional pluri-Lagrangian system generated by a discrete three-point 2-form. The commutativity proof is based on the construction of the so called *local superposition formulae* which turn out to be nothing but the suitably interpreted 3D corner equations. Moreover, these superposition formulae enable us to handle the multi-valuedness of Bäcklund transformations in the case of periodic boundary conditions, by means of a precise description of the branching behavior of the multi-valued maps.
- In sections 4–8, these general results are applied to all systems of the relativistic Toda type, as listed in [Adl99, Sur03]. For each of the systems, we identify all the ingredients of the pluri-Lagrangian structure, which allows us to give unified proofs for commutativity of all maps in question. In particular, in all cases we prove the so-called *closure relation*, which expresses the fact that the Lagrangian 1-form on the multi-time (space of independent variables) is closed on solutions of variational equations, and turns out to be the main feature of the Lagrangian theory. These results generalize the ones from our recent paper [BPS13]: sending the relativistic parameter α to 0, we recover the corresponding results for the non-relativistic case obtained in [BPS13]. This reinforces the observation already made in [BS10b]: the non-relativistic degeneration obscures the natural relations to two-dimensional lattice systems.
- Finally, in section 9, we turn to the question of the relation of the pluri-Lagrangian structure to more traditional attributes and notions of integrability. In the one-dimensional context,

such a relation is established through connecting the closure relation with the *spectrality property*, introduced by Kuznetsov and Sklyanin [KS98], which says that the derivative of the Lagrangian with respect to the parameter of a family of Bäcklund transformations is a generating function of common integrals of motion for the whole family. In the two-dimensional context, we establish a new result which connects the closure relation with a parameter-dependent family of *local conservation laws*. Again, the densities of these conservation laws turn out to be composed of the derivatives of the Lagrangian 2-form with respect to the Bäcklund parameter. Moreover, in the framework of the relativistic Toda-type systems, these conservation laws turn out to constitute a local form of the integrals of motion provided by the spectrality property.

2 General theory of discrete pluri-Lagrangian systems

Definition 2.1 (*d*-dimensional pluri-Lagrangian problem). Let \mathcal{L} be a discrete *d*-form on \mathbb{Z}^m , depending on some field $x : \mathbb{Z}^m \rightarrow \mathcal{X}$, where \mathcal{X} is some vector space.

- To an arbitrary oriented *d*-dimensional manifold Σ in \mathbb{Z}^m , there corresponds the *action functional*, which assigns to $x|_{V(\Sigma)}$, i.e., to the fields at the vertices of Σ , the number

$$S_\Sigma = \sum_{\sigma \in \Sigma} \mathcal{L}(\sigma).$$

- We say that the field $x : V(\Sigma) \rightarrow \mathcal{X}$ is a *critical point* of S_Σ , if at any interior point $n \in V(\Sigma)$, we have

$$\frac{\partial S_\Sigma}{\partial x(n)} = 0.$$

- We say that the field $x : \mathbb{Z}^m \rightarrow \mathcal{X}$ solves the *pluri-Lagrangian problem* for the Lagrangian *d*-form \mathcal{L} if, for any oriented *d*-dimensional manifold Σ in \mathbb{Z}^m , the restriction $x|_{V(\Sigma)}$ is a critical point of the corresponding action S_Σ .

2.1 One-dimensional pluri-Lagrangian systems, $d = 1$

This section is based on [Sur13].

In the case $d = 1$, \mathcal{L} is a function of directed edges σ of \mathbb{Z}^m with $\mathcal{L}(-\sigma) = -\mathcal{L}(\sigma)$. Thus,

$$\mathcal{L}(\sigma_i) = \mathcal{L}(n, n + e_i) = \Lambda_i(x, x_i) \quad \Leftrightarrow \quad \mathcal{L}(-\sigma_i) = \mathcal{L}(n + e_i, n) = -\Lambda_i(x, x_i).$$

Here $\Lambda_i : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ are local Lagrangian functions corresponding to the edges of the i^{th} coordinate direction, and the following abbreviations are used: x for $x(n)$ at a generic point $n \in \mathbb{Z}^m$, and then

$$x_i = x(n + e_i), \quad x_{-i} = x(n - e_i), \quad i = 1, \dots, m, \quad (1)$$

where e_i is the unit vector of the i^{th} coordinate direction.

Any interior point of any discrete curve Σ in \mathbb{Z}^m is of one of the four types shown on Figure 1.

The pieces of discrete curves as on Figures 1(b), (c), and (d) will be called *2D corners*. Observe that a straight piece of a discrete curve, as on Figure 1(a) is a sum of 2D corners, as on Figures 1(b)

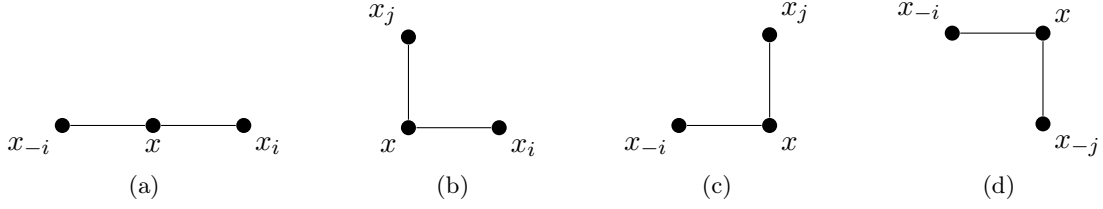


Figure 1: Four type of vertices of a discrete curve. Case (a): two edges of one coordinate direction meet at n . Case (b): a negatively directed edge followed by a positively directed edge. Case (c): two equally (positively or negatively) directed edges of two different coordinate directions meet at n . Case (d): a positively directed edge followed by a negatively directed edge.

and (c). The whole variety of Euler-Lagrange equations for a pluri-Lagrangian system with $d = 1$ reduces to the following three types of $2D$ corner equations:

$$\frac{\partial \Lambda_i(x, x_i)}{\partial x} - \frac{\partial \Lambda_j(x, x_j)}{\partial x} = 0, \quad (2)$$

$$\frac{\partial \Lambda_i(x_{-i}, x)}{\partial x} + \frac{\partial \Lambda_j(x, x_j)}{\partial x} = 0, \quad (3)$$

$$\frac{\partial \Lambda_i(x_{-i}, x)}{\partial x} - \frac{\partial \Lambda_j(x_{-j}, x)}{\partial x} = 0. \quad (4)$$

In particular, the standard single-time discrete Euler-Lagrange equation,

$$\frac{\partial \Lambda_i(x_{-i}, x)}{\partial x} + \frac{\partial \Lambda_i(x, x_i)}{\partial x} = 0,$$

corresponding to a straight piece of a discrete curve as on Figure 1(a) is a consequence of equations (2) and (3), corresponding to 2D corners as on on Figures 1(b) and (c).

To discuss the *consistency* of the system of 2D corner equations, it will be more convenient to re-write them with appropriate shifts, as

$$\frac{\partial \Lambda_i(x, x_i)}{\partial x} - \frac{\partial \Lambda_j(x, x_j)}{\partial x} = 0, \quad (E)$$

$$\frac{\partial \Lambda_i(x, x_i)}{\partial x_i} + \frac{\partial \Lambda_j(x_i, x_{ij})}{\partial x_i} = 0, \quad (E_i)$$

$$\frac{\partial \Lambda_j(x, x_j)}{\partial x_j} + \frac{\partial \Lambda_i(x_j, x_{ij})}{\partial x_j} = 0, \quad (E_j)$$

$$\frac{\partial \Lambda_i(x_j, x_{ij})}{\partial x_{ij}} - \frac{\partial \Lambda_j(x_i, x_{ij})}{\partial x_{ij}} = 0. \quad (E_{ij})$$

In this form, 2D corner equations (E)–(E_{ij}) correspond to the four vertices of an elementary square σ_{ij} of the lattice, as on Figure 2(a). Consistency of the system of 2D corner equations (E)–(E_{ij}) should be understood as follows: start with the fields x, x_i, x_j satisfying equation (E). Then each of equations (E_i), (E_j) can be solved for x_{ij} . Thus, we obtain two alternative values for the latter field. Consistency takes place if these values coincide identically (with respect to the initial data), and, moreover, if the resulting field x_{ij} satisfies equation (E_{ij}). In other words:

Definition 2.2. The system of 2D corner equations (E)–(E_{ij}) is called *consistent*, if it has the minimal possible rank 2, i.e., if exactly two of these four equations are independent.

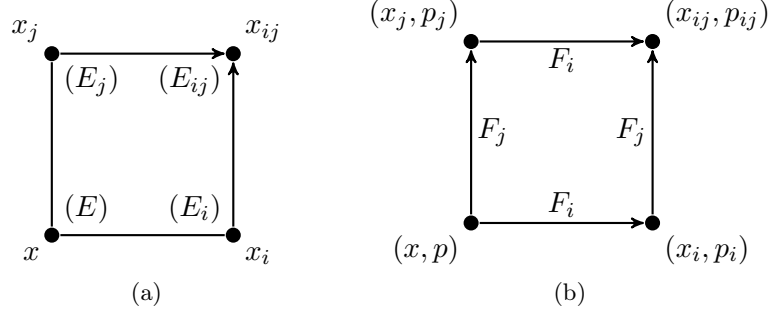


Figure 2: Consistency of 2D corner equations: (a) Start with data x, x_i, x_j related by 2D corner equation (E); solve 2D corner equations (E_i) and (E_j) for x_{ij} ; consistency means that the two values of x_{ij} coincide identically and satisfy 2D corner equation (E_{ij}). (b) Maps F_i and F_j commute.

Observe that 2D corner equations (E)–(E_{ij}) can be put as

$$\frac{\partial S^{ij}}{\partial x} = 0, \quad \frac{\partial S^{ij}}{\partial x_i} = 0, \quad \frac{\partial S^{ij}}{\partial x_j} = 0, \quad \frac{\partial S^{ij}}{\partial x_{ij}} = 0, \quad (5)$$

where S^{ij} is the action along the boundary of an oriented elementary square σ_{ij} (this action can be identified with the discrete exterior derivative $d\mathcal{L}$ evaluated at σ_{ij}),

$$S^{ij} = d\mathcal{L}(\sigma_{ij}) = \Delta_i \mathcal{L}(\sigma_j) - \Delta_j \mathcal{L}(\sigma_i) = \Lambda_i(x, x_i) + \Lambda_j(x_i, x_{ij}) - \Lambda_i(x_j, x_{ij}) - \Lambda_j(x, x_j).$$

The main feature of our definition is that the “almost closedness” of the 1-form \mathcal{L} on solutions of the system of 2D corner equations is, so to say, built-in from the outset.

Theorem 2.3. *For any pair of the coordinate directions i, j , the action S^{ij} over the boundary of an elementary square of these coordinate directions is constant on solutions of the system of 2D corner equations (11):*

$$S^{ij}(x, x_i, x_{ij}, x_j) = \ell^{ij} = \text{const} \pmod{\partial S^{ij}/\partial x = 0, \dots, \partial S^{ij}/\partial x_{ij} = 0}.$$

In particular, if all these constants ℓ^{ij} vanish, then the discrete 1-form \mathcal{L} is closed on solutions of the Euler-Lagrange equations, so that the critical value of the action functional S_Σ does not depend on the choice of the curve Σ connecting two given points in \mathbb{Z}^m .

We now turn to the Hamiltonian part of the theory of one-dimensional pluri-Lagrangian systems. Consistency of the system of 2D corner equations (2)–(4) is equivalent to existence of a function $p : \mathbb{Z}^m \rightarrow \mathcal{X}$ satisfying all the relations

$$p = \frac{\partial \Lambda_i(x, x_i)}{\partial x}, \quad i = 1, \dots, m, \quad (6)$$

$$p = -\frac{\partial \Lambda_i(x_{-i}, x)}{\partial x}, \quad i = 1, \dots, m. \quad (7)$$

We say that the multi-time discrete Lagrangian 1-form \mathcal{L} is *Legendre transformable*, if all the equations (6) can be solved for x_i in terms of x, p . In this case, equations

$$p = \frac{\partial \Lambda_i(x, x_i)}{\partial x}, \quad p_i = -\frac{\partial \Lambda_i(x, x_i)}{\partial x_i}, \quad (8)$$

define a symplectic map $F_i : (x, p) \mapsto (x_i, p_i)$.

Theorem 2.4. *For a consistent one-dimensional pluri-Lagrangian system with a Legendre-transformable 1-form \mathcal{L} , maps F_i commute:*

$$F_i \circ F_j = F_j \circ F_i, \quad (9)$$

see Figure 2(b)). Conversely, for a given system of m commuting symplectic maps F_i admitting Lagrangians (generating functions) Λ_i , the 1-form \mathcal{L} defined by $\mathcal{L}(\sigma_i) = \Lambda_i(x, x_i)$, generates a consistent one-dimensional pluri-Lagrangian system.

2.2 Two-dimensional pluri-Lagrangian systems, $d = 2$

This section is based on [BPS14a].

In the case $d = 2$, \mathcal{L} is a function of oriented elementary squares

$$\sigma_{ij} = (n, n + e_i, n + e_i + e_j, n + e_j),$$

such that $\mathcal{L}(\sigma_{ij}) = -\mathcal{L}(\sigma_{ji})$.

One can show that the flower of any interior vertex of an oriented quad-surface Σ in \mathbb{Z}^m can be represented as a sum of (oriented) 3D corners in \mathbb{Z}^{m+1} . Here, a *3D corner* is a quad-surface consisting of three elementary squares adjacent to a vertex of valence 3. Examples of 3D corners are given in [BPS14a]. As a consequence, the action for any flower can be represented as a sum of actions for several 3D corners. Thus, Euler-Lagrange equation for any interior vertex n of Σ can be represented as a sum of several Euler-Lagrange equations for 3D corners. This justifies the following fundamental definition:

Definition 2.5. The *system of 3D corner equations* for a given discrete 2-form \mathcal{L} consists of discrete Euler-Lagrange equations for all possible 3D corners in \mathbb{Z}^m . If the action for the surface of an oriented elementary cube σ_{ijk} of the coordinate directions i, j, k (which can be identified with the discrete exterior derivative $d\mathcal{L}$ evaluated at σ_{ijk}) is denoted by

$$S^{ijk} = d\mathcal{L}(\sigma_{ijk}) = \Delta_k \mathcal{L}(\sigma_{ij}) + \Delta_i \mathcal{L}(\sigma_{jk}) + \Delta_j \mathcal{L}(\sigma_{ki}), \quad (10)$$

then the system of 3D corner equations consists of the eight equations

$$\begin{aligned} \frac{\partial S^{ijk}}{\partial x} &= 0, & \frac{\partial S^{ijk}}{\partial x_i} &= 0, & \frac{\partial S^{ijk}}{\partial x_j} &= 0, & \frac{\partial S^{ijk}}{\partial x_k} &= 0, \\ \frac{\partial S^{ijk}}{\partial x_{ij}} &= 0, & \frac{\partial S^{ijk}}{\partial x_{jk}} &= 0, & \frac{\partial S^{ijk}}{\partial x_{ik}} &= 0, & \frac{\partial S^{ijk}}{\partial x_{ijk}} &= 0, \end{aligned} \quad (11)$$

for each triple i, j, k .

Thus, the system of 3D corner equations encompasses all possible discrete Euler-Lagrange equations for all possible quad-surfaces Σ . In other words, solutions of a two-dimensional pluri-Lagrangian problem as introduced in Definition 2.1 are precisely solutions of the corresponding system of 3D corner equations.

Remark. We formulated the system of 3D corner equations for a generic 2-form \mathcal{L} . In particular cases the quantity S^{ijk} could be independent on some of the fields at the corners of the cube. Then the system of 3D corner equations (11) could contain less equations.

Of course, in order that the above definition be meaningful, the system of 3D corner equations has to be *consistent*:

Definition 2.6. The system (11) is called *consistent*, if it has the minimal possible rank 2, i.e., if exactly two of these equations are independent.

Again, the “almost closedness” of the 2-form \mathcal{L} on solutions of the system of 3D corner equations is built-in from the outset.

Theorem 2.7. For any triple of the coordinate directions i, j, k , the action S^{ijk} over an elementary cube of these coordinate directions is constant on solutions of the system of 3D corner equations (11):

$$S^{ijk}(x, \dots, x_{ijk}) = c^{ijk} = \text{const} \pmod{\partial S^{ijk}/\partial x = 0, \dots, \partial S^{ijk}/\partial x_{ijk} = 0}.$$

The most interesting case is, of course, when all $c^{ijk} = 0$. Then $d\mathcal{L} = 0$, that is, the discrete 2-form \mathcal{L} is *closed* on solutions of the system of 3D corner equations, so that the critical value of the action S_Σ does not change under perturbations of the quad-surface Σ in \mathbb{Z}^m fixing its boundary.

3 From 2D pluri-Lagrangian systems to relativistic Toda type systems

We start with a general 3-point 2-form

$$\mathcal{L}(\sigma_{ij}) = L_i(x_i - x) - L_j(x_j - x) - \Lambda_{ij}(x_j - x_i), \quad (12)$$

where the Lagrangians L_i and Λ_{ij} only depend on the differences of the fields at the end points, and the diagonal Lagrangians are skew-symmetric in the sense that $\Lambda_{ij}(x) = -\Lambda_{ji}(-x)$.

For a 3-point 2-form, expression (10) specializes to

$$\begin{aligned} S^{ijk} = & L_i(x_{ik} - x_k) + L_j(x_{ij} - x_i) + L_k(x_{jk} - x_j) \\ & - L_i(x_{ij} - x_j) - L_j(x_{jk} - x_k) - L_k(x_{ik} - x_i) \\ & - \Lambda_{ij}(x_{jk} - x_{ik}) - \Lambda_{jk}(x_{ik} - x_{ij}) - \Lambda_{ki}(x_{ij} - x_{jk}) \\ & + \Lambda_{ij}(x_j - x_i) + \Lambda_{jk}(x_k - x_j) + \Lambda_{ki}(x_i - x_k). \end{aligned} \quad (13)$$

Thus, S^{ijk} depends on neither x nor x_{ijk} , and its domain of definition is better visualized as an octahedron shown in Figure 3.

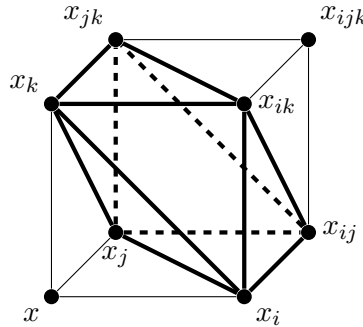


Figure 3: Octahedron supporting $d\mathcal{L}$ for a 3-point discrete 2-form \mathcal{L}

Accordingly, the system of corner equations consists of six equations per elementary 3D cube, which we denote by (\mathcal{E}_i) , (\mathcal{E}_j) , (\mathcal{E}_k) , (\mathcal{E}_{ij}) , (\mathcal{E}_{ik}) , and (\mathcal{E}_{jk}) . Our main assumption is that *the system of corner equations is consistent*. To write them down, we set

$$\psi_i(x) = \frac{\partial L_i(x)}{\partial x}, \quad \phi_{ij}(x) = \frac{\partial \Lambda_{ij}(x)}{\partial x}. \quad (14)$$

In particular, we have: $\phi_{ij}(x) = \phi_{ji}(-x)$. In terms of these functions, corner equations read:

$$\psi_j(x_{ij} - x_i) + \phi_{ij}(x_j - x_i) = \psi_k(x_{ik} - x_i) + \phi_{ik}(x_k - x_i), \quad (\mathcal{E}_i)$$

$$\psi_j(x_{ij} - x_i) + \phi_{kj}(x_{ij} - x_{ik}) = \psi_i(x_{ij} - x_j) + \phi_{ki}(x_{ij} - x_{jk}). \quad (\mathcal{E}_{ij})$$

In what follows, one of the coordinate directions (which we denote as the 0^{th} one) plays a distinguished role, it enumerates the sites of the relativistic Toda chains. We will use the index n for this coordinate direction only. Accordingly, we will only consider surfaces in \mathbb{Z}^m which contain, along with any point, the whole line through this point parallel to the 0^{th} coordinate axis. One can call such surfaces *cylindrical*. The set of values of x along such a line, $x = \{x_n : n \in \mathbb{Z}\}$, or, upon a finite-dimensional reduction, $x = \{x_n : 1 \leq n \leq N\}$, is an element of the configuration space of the relativistic Toda lattice. We use the accents $\tilde{}$ and $\hat{}$ to denote the shift in the discrete times corresponding to all other coordinate directions.

Definition 3.1. The map $F_i : (x, p) \mapsto (\tilde{x}, \tilde{p})$ is the symplectic map with the generating function

$$\mathfrak{L}_i(x, \tilde{x}) = \sum_{n=1}^N L_i(\tilde{x}_n - x_n) - \sum_{n=1}^N L_0(x_{n+1} - x_n) - \sum_{n=1}^N \Lambda_{i0}(x_{n+1} - \tilde{x}_n), \quad (15)$$

thus its equations of motion $p_n = -\partial \mathfrak{L}_i / \partial x_n$, $\tilde{p}_n = \partial \mathfrak{L}_i / \partial \tilde{x}_n$ read:

$$F_i : \begin{cases} p_n = \psi_i(\tilde{x}_n - x_n) + \phi_{i0}(x_n - \tilde{x}_{n-1}) - \psi_0(x_{n+1} - x_n) + \psi_0(x_n - x_{n-1}), \\ \tilde{p}_n = \psi_i(\tilde{x}_n - x_n) + \phi_{i0}(x_{n+1} - \tilde{x}_n). \end{cases} \quad (16)$$

The Euler-Lagrange equations read

$$\psi_i(\tilde{x}_n - x_n) - \psi_i(x_n - \tilde{x}_n) = \psi_0(x_{n+1} - x_n) - \psi_0(x_n - x_{n-1}) + \phi_{i0}(\tilde{x}_{n+1} - x_n) - \phi_{i0}(x_n - \tilde{x}_{n-1}). \quad (17)$$

This map corresponds to the edges $(x, \tilde{x}) = (x, x_i)$ of the i^{th} coordinate direction, to which the strip supporting \mathfrak{L}_i projects along the 0^{th} coordinate axis. See the identifications of variables on Figure 4. We denote the index set of the maps F_i by $I = \{i, j, \dots\}$.

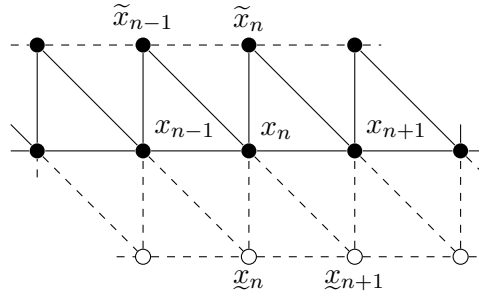


Figure 4: Domain of the map F_i

Definition 3.2. The map $G_k : (x, p) \mapsto (\tilde{x}, \tilde{p})$ is the symplectic map with the generating function

$$\mathfrak{M}_k(x, \tilde{x}) = \sum_{n=1}^N \Lambda_{k0}(\tilde{x}_n - x_n) + \sum_{n=1}^N L_0(\tilde{x}_n - \tilde{x}_{n-1}) - \sum_{n=1}^N L_k(x_n - \tilde{x}_{n-1}), \quad (18)$$

thus its equations of motion $p_n = -\partial \mathfrak{M}_k / \partial x_n$, $\tilde{p}_n = \partial \mathfrak{M}_k / \partial \tilde{x}_n$ read:

$$G_k : \begin{cases} p_n = \phi_{k0}(\tilde{x}_n - x_n) + \psi_k(x_n - \tilde{x}_{n-1}), \\ \tilde{p}_n = \phi_{k0}(\tilde{x}_n - x_n) + \psi_k(x_{n+1} - \tilde{x}_n) - \psi_0(\tilde{x}_{n+1} - \tilde{x}_n) + \psi_0(\tilde{x}_n - \tilde{x}_{n-1}). \end{cases} \quad (19)$$

The Euler-Lagrange equations read

$$\phi_{k0}(\tilde{x}_n - x_n) - \phi_{k0}(x_n - \tilde{x}_n) = -\psi_0(x_{n+1} - x_n) + \psi_0(x_n - x_{n-1}) + \psi_k(\tilde{x}_{n+1} - x_n) - \psi_j(x_n - \tilde{x}_{n-1}). \quad (20)$$

This map corresponds to the negatively directed edges $(x, \tilde{x}) = (x, x_{-k})$ of the k^{th} coordinate direction, to which the strip supporting \mathfrak{M}_k projects along the 0^{th} coordinate axis. See the identifications of variables on Figure 5. We denote the index set of the maps G_k by $K = \{k, \ell, \dots\}$, and assume it to be disjoint from $I = \{i, j, \dots\}$.

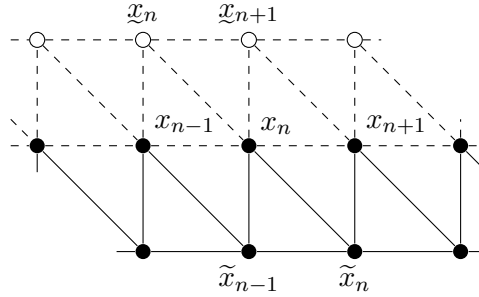


Figure 5: Domain of the map G_k

In the present paper, we consider maps F_i , G_k with finitely many degrees of freedom ($1 \leq n \leq N$). This requires to specify certain boundary conditions. We will consider either the so-called open-end or periodic boundary conditions.

- Open-end boundary conditions correspond to letting the second and the third sums in the Lagrangian functions (15), (18) extend over $1 \leq n \leq N-1$ only. Effectively, this amounts to omitting terms containing x_0 or \tilde{x}_0 from the expressions for p_1 and \tilde{p}_1 , and likewise omitting terms containing x_{N+1} or \tilde{x}_{N+1} from the expressions for p_N and \tilde{p}_N . In this case, maps F_i and G_k are *single-valued* functions of (x, p) .
- Periodic boundary conditions correspond to letting all indices be taken mod N , so that $x_0 = x_N$, $x_{N+1} = x_1$. In this case, these maps are *double-valued*, so that the very notion of their commutativity has to be clarified. We achieve this along the same lines as in the previous work [BPS13].

Theorem 3.3. Let i, j, k , and ℓ be four different indices from $I = \{i, j, \dots\}$ and $K = \{k, \ell, \dots\}$. Then any two of the maps F_i , F_j , G_k , and G_ℓ commute.

The next three subsections are devoted to the proof of this theorem. We prove separately the commutativity of F_i and F_j , of G_k and G_ℓ , and of F_i and G_ℓ . As explained in Section 2.1, each such statement is equivalent to consistency of the corresponding system of 2D corner equations.

3.1 Proof of commutativity of the maps F_i, F_j

The 2D corner equations for the pluri-Lagrangian system corresponding to two maps F_i and F_j read:

$$\psi_i(\tilde{x}_n - x_n) + \phi_{i0}(x_n - \tilde{x}_{n-1}) = \psi_j(\hat{x}_n - x_n) + \phi_{j0}(x_n - \hat{x}_{n-1}), \quad (E)$$

$$\begin{aligned} \psi_i(\tilde{x}_n - x_n) + \phi_{i0}(x_{n+1} - \tilde{x}_n) &= \psi_j(\tilde{\hat{x}}_n - \tilde{x}_n) + \phi_{j0}(\tilde{x}_n - \tilde{\hat{x}}_{n-1}) \\ &\quad - \psi_0(\tilde{x}_{n+1} - \tilde{x}_n) + \psi_0(\tilde{x}_n - \tilde{x}_{n-1}), \end{aligned} \quad (E_i)$$

$$\begin{aligned} \psi_j(\hat{x}_n - x_n) + \phi_{j0}(x_{n+1} - \hat{x}_n) &= \psi_i(\hat{\tilde{x}}_n - \hat{x}_n) + \phi_{i0}(\hat{x}_n - \hat{\tilde{x}}_{n-1}) \\ &\quad - \psi_0(\hat{x}_{n+1} - \hat{x}_n) + \psi_0(\hat{x}_n - \hat{x}_{n-1}), \end{aligned} \quad (E_j)$$

$$\psi_i(\hat{\tilde{x}}_n - \hat{x}_n) + \phi_{i0}(\hat{x}_{n+1} - \hat{\tilde{x}}_n) = \psi_j(\tilde{\hat{x}}_n - \tilde{x}_n) + \phi_{j0}(\tilde{x}_{n+1} - \tilde{\hat{x}}_n). \quad (E_{ij})$$

A visualization of the 2D corner equations embedded in \mathbb{Z}^3 is given in Figure 6. Discrete curves

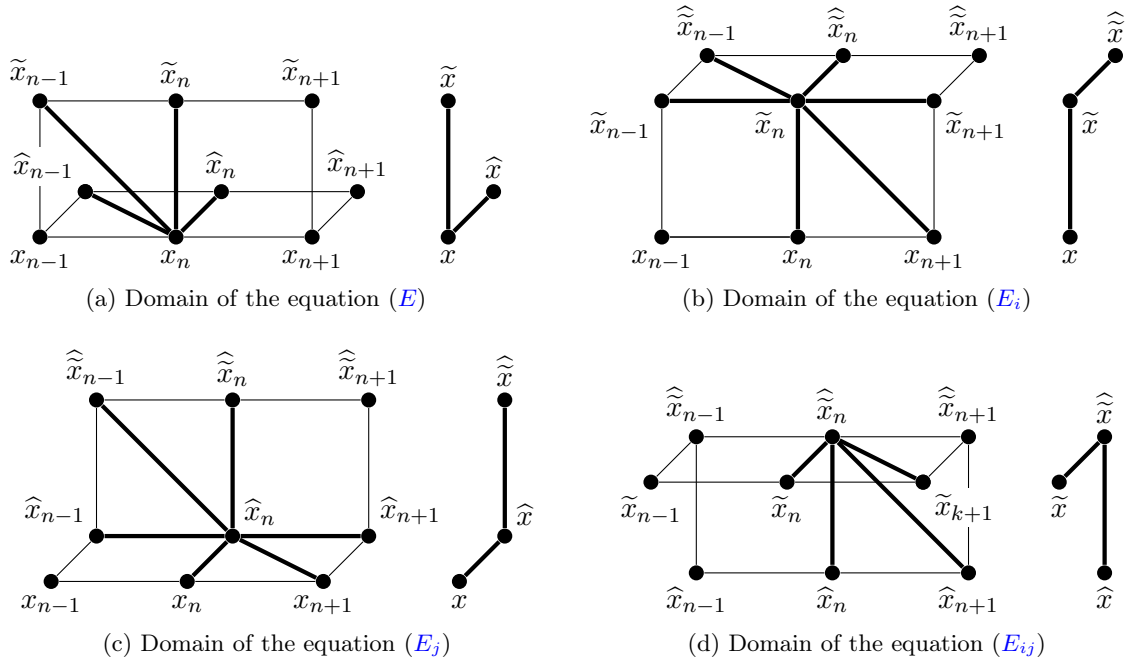


Figure 6: 2D corner equations for the system of F_i and F_j : equations (E) and (E_{ij}) are 3D corner equations of the corresponding discrete 2-form, while equations (E_i) and (E_j) are sums of two 3D corner equations coming from two 3D corners with one common face.

in the multi-time plane \mathbb{Z}^2 are in a one-to-one correspondence with cylindrical surfaces in \mathbb{Z}^3 , via the projection along the first coordinate direction of \mathbb{Z}^3 . For 2D corners, this is illustrated in Figure 6.

Consistency of the above system of 2D corner equations is proven with the help of the following statement.

Theorem 3.4. *Suppose that the fields x, \tilde{x} , and \hat{x} satisfy 2D corner equations (E) . Define the*

fields $\widehat{\tilde{x}}$ by any of the following four formulae, which are equivalent by virtue of (E):

$$\psi_j(\widehat{\tilde{x}}_n - \tilde{x}_n) + \phi_{ij}(\widehat{\tilde{x}}_n - \tilde{x}_n) = \psi_0(\tilde{x}_{n+1} - \tilde{x}_n) + \phi_{i0}(x_{n+1} - \tilde{x}_n), \quad (S1)$$

$$\psi_i(\widehat{\tilde{x}}_n - \widehat{x}_n) + \phi_{ji}(\tilde{x}_n - \widehat{x}_n) = \psi_0(\widehat{x}_{n+1} - \widehat{x}_n) + \phi_{j0}(x_{n+1} - \widehat{x}_n), \quad (S2)$$

$$\psi_i(\tilde{x}_{n+1} - x_{n+1}) + \phi_{ji}(\tilde{x}_{n+1} - \widehat{x}_{n+1}) = \psi_0(\tilde{x}_{n+1} - \tilde{x}_n) + \phi_{j0}(\tilde{x}_{n+1} - \widehat{\tilde{x}}_n), \quad (S3)$$

$$\psi_j(\widehat{x}_{n+1} - x_{n+1}) + \phi_{ij}(\widehat{x}_{n+1} - \tilde{x}_{n+1}) = \psi_0(\widehat{x}_{n+1} - \widehat{x}_n) + \phi_{i0}(\widehat{x}_{n+1} - \widehat{\tilde{x}}_n), \quad (S4)$$

called superposition formulae (note that each one of these formulae is local with respect to $\widehat{\tilde{x}}$). Then the 2D corner equations (E_i), (E_j), and (E_{ij}) are satisfied, as well.

Proof. Consider the sublattice \mathbb{Z}^3 spanned by the coordinate directions 0 (indexed by the letter n), i , corresponding to the map F_i (shift in this direction being denoted by \sim), and j , corresponding to the map F_j (shift in this direction being denoted by $\widehat{}$). See Figure 7.

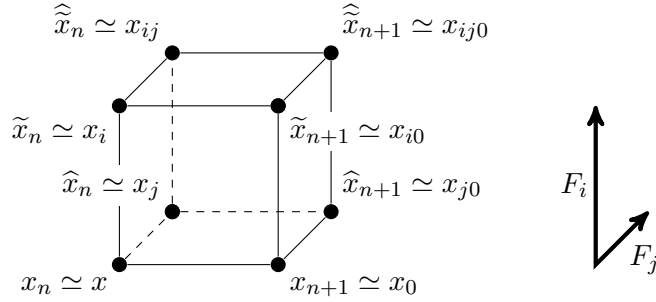


Figure 7: Identification of fields: the 0th coordinate direction with the space-direction, the i^{th} coordinate direction with the time-direction of the map F_i and the j^{th} direction with the time-direction of the map F_j .

One easily checks that, upon the identifications as on Figure 7, the two corner equations (E), (E_{ij}) and the four superposition formulae (S1)–(S4) build nothing but the system of 3D corner equations (E_i), (E_j). Due to consistency of the latter system, as formulated in Theorem 3.4, if equation (E) and one of equations (S1)–(S4) hold, then equation (E_{ij}) and the remaining three of equations (S1)–(S4) are satisfied, as well. Furthermore, equation (E_i) is the difference of (S1) and the downshifted version of (S3), while equation (E_j) is the difference of (S2) and the downshifted version of (S4). This completes the proof. \square

This theorem provides us with an exhaustive understanding of commutativity of double-valued Bäcklund transformations in the periodic case:

- In the Lagrangian picture, suppose that we are given fields $x, \tilde{x}, \widehat{x}$ satisfying the 2D corner equation (E). Each of equations (E_i), (E_j) produces two values for $\widehat{\tilde{x}}$. Consistency is reflected in the following fact: one of the values for $\widehat{\tilde{x}}$ obtained from (E_i) coincides with one of the values for $\widehat{\tilde{x}}$ obtained from (E_j). Indeed, this common value is nothing but $\widehat{\tilde{x}}$ obtained from the superposition formulae (S1), (S2), (S3) or (S4), as in Theorem 3.4.
- In the symplectic maps picture, each of the compositions $F_i \circ F_j$ and $F_j \circ F_i$ applied to a point (x, p) produces four different branches for (\tilde{x}, \widehat{p}) . Commutativity is reflected in the following fact: each of the branches of $F_i \circ F_j$ coincides with one of the branches of $F_j \circ F_i$. Indeed, Theorem 3.4 delivers four possible values for $(\tilde{x}, \widehat{x}, \widehat{\tilde{x}})$ satisfying all 2D corner equations (E)–(E_{ij}), namely one $\widehat{\tilde{x}}$ for each of the four possible combinations of (\tilde{x}, \widehat{x}) .

The reader is referred to [BPS13] for a graphical illustration and more details.

3.2 Proof of commutativity of the maps G_k, G_ℓ

The 2D corner equations for the pluri-Lagrangian system corresponding to the two maps G_k and G_ℓ read:

$$\phi_{k0}(\tilde{x}_n - x_n) + \psi_k(x_n - \tilde{x}_{n-1}) = \phi_{\ell 0}(\hat{x}_n - x_n) + \psi_\ell(x_n - \hat{x}_{n-1}), \quad (E)$$

$$\begin{aligned} \phi_{k0}(\tilde{x}_n - x_n) + \psi_k(x_{n+1} - \tilde{x}_n) - \psi_0(\tilde{x}_{n+1} - \tilde{x}_n) + \psi_0(\tilde{x}_n - \tilde{x}_{n-1}) \\ = \phi_{\ell 0}(\hat{x}_n - \tilde{x}_n) + \psi_\ell(\tilde{x}_n - \hat{x}_{n-1}), \end{aligned} \quad (E_k)$$

$$\begin{aligned} \phi_{\ell 0}(\hat{x}_n - x_n) + \psi_\ell(x_{n+1} - \hat{x}_n) - \psi_0(\hat{x}_{n+1} - \hat{x}_n) + \psi_0(\hat{x}_n - \hat{x}_{n-1}) \\ = \phi_{k0}(\hat{x}_n - \tilde{x}_n) + \psi_k(\hat{x}_n - \hat{x}_{n-1}), \end{aligned} \quad (E_\ell)$$

$$\phi_{k0}(\hat{x}_n - \tilde{x}_n) + \psi_k(\hat{x}_{n+1} - \hat{x}_n) = \phi_{\ell 0}(\hat{x}_n - \tilde{x}_n) + \psi_\ell(\tilde{x}_{n+1} - \tilde{x}_n). \quad (E_{k\ell})$$

A visualization of the 2D corner equations embedded in \mathbb{Z}^3 is given in Figure 8.

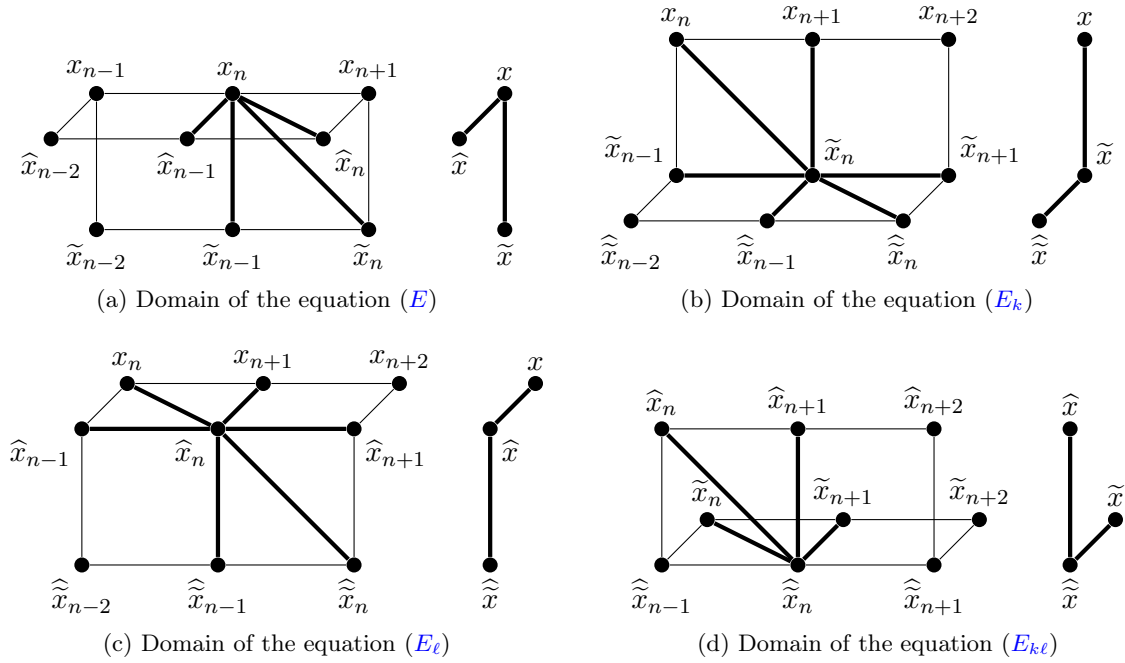


Figure 8: 2D corner equations for commutativity of G_k and G_ℓ : 2D corner equations (E), (E_{kℓ}) are 3D corner equations of the corresponding discrete 2-form, while each of the 2D corner equations (E_k), (E_ℓ) is a sum of two 3D corner equations sharing one common face.

Theorem 3.5. Suppose that the fields x , \tilde{x} , and \hat{x} satisfy 2D corner equations (E). Define the fields $\hat{\tilde{x}}$ by any of the following four formulae, which are equivalent by virtue of (E):

$$\psi_k(x_{n+1} - \tilde{x}_n) + \phi_{\ell k}(\hat{x}_n - \tilde{x}_n) = \psi_0(\tilde{x}_{n+1} - \tilde{x}_n) + \phi_{\ell 0}(\hat{x}_n - \tilde{x}_n), \quad (S1)$$

$$\psi_\ell(x_{n+1} - \hat{x}_n) + \phi_{k\ell}(\tilde{x}_n - \hat{x}_n) = \psi_0(\hat{x}_{n+1} - \hat{x}_n) + \phi_{k0}(\tilde{x}_n - \hat{x}_n), \quad (S2)$$

$$\psi_k(\hat{x}_n - \hat{\tilde{x}}_{n-1}) + \phi_{\ell k}(\hat{x}_n - \tilde{x}_n) = \psi_0(\hat{x}_n - \hat{x}_{n-1}) + \phi_{\ell 0}(\hat{x}_n - x_n), \quad (S3)$$

$$\psi_\ell(\tilde{x}_n - \hat{\tilde{x}}_{n-1}) + \phi_{k\ell}(\tilde{x}_n - \hat{x}_n) = \psi_0(\tilde{x}_n - \tilde{x}_{n-1}) + \phi_{k0}(\tilde{x}_n - x_n). \quad (S4)$$

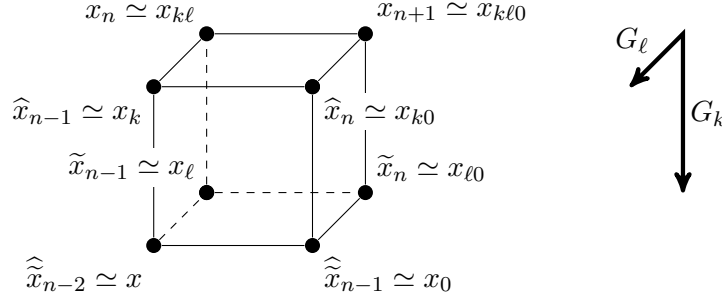


Figure 9: Identification of fields: the 0th coordinate direction enumerates the lattice cites, the coordinate directions k, ℓ correspond to the maps G_k, G_ℓ .

called *superposition formulae*. Then 2D corner equations (E_k) – $(E_{k\ell})$ are satisfied, as well.

Proof. We identify the fields as on Figure 9. Then equations (E) , $(E_{k\ell})$ and $(S1)$ – $(S3)$ build the system of consistent 3D corner equations (\mathcal{E}_i) , (\mathcal{E}_{ij}) . More precisely, the correspondence is as follows:

- The downshifted version of (E) is $(\mathcal{E}_{k\ell})$.
- The downshifted version of $(E_{k\ell})$ is (\mathcal{E}_0) .
- The downshifted versions of $(S1)$ and $(S2)$ are (\mathcal{E}_ℓ) and (\mathcal{E}_k) , respectively.
- Equations $(S4)$ and $(S3)$ are $(\mathcal{E}_{0\ell})$ and (\mathcal{E}_{0k}) , respectively.

Since the system of 3D corner equations is consistent and, therefore, has rank 2, the following argumentation works: if (E) and one of equations $(S1)$ – $(S3)$ are satisfied, then equation $(E_{k\ell})$ and the remaining three equations of $(S1)$ – $(S3)$ are fulfilled, as well. Furthermore, equation (E_k) is a difference of $(S1)$ and $(S3)$, and equation (E_ℓ) is a difference of $(S2)$ and $(S4)$. This completes the proof. \square

3.3 Proof of commutativity of the maps F_i, G_ℓ

The 2D corner equations for the pluri-Lagrangian system corresponding to the maps F_i and G_ℓ (whose actions are encoded by \sim and $\hat{\cdot}$, respectively) are given by:

$$\begin{aligned} \psi_i(\tilde{x}_n - x_n) + \phi_{i0}(x_n - \tilde{x}_{n-1}) - \psi_0(x_{n+1} - x_n) + \psi_0(x_n - x_{n-1}) \\ = \phi_{\ell 0}(\hat{x}_n - x_n) + \psi_\ell(x_n - \hat{x}_{n-1}), \end{aligned} \quad (E)$$

$$\psi_i(\tilde{x}_n - x_n) + \phi_{i0}(x_{n+1} - \tilde{x}_n) = \phi_{\ell 0}(\hat{\tilde{x}}_n - \tilde{x}_n) + \psi_\ell(\tilde{x}_n - \hat{\tilde{x}}_{n-1}), \quad (E_i)$$

$$\psi_i(\hat{\tilde{x}}_n - \hat{x}_n) + \phi_{i0}(\hat{x}_n - \hat{\tilde{x}}_{n-1}) = \phi_{\ell 0}(\hat{x}_n - x_n) + \psi_\ell(x_{n+1} - \hat{x}_n), \quad (E_\ell)$$

$$\begin{aligned} \psi_i(\hat{\tilde{x}}_n - \hat{x}_n) + \phi_{i0}(\hat{x}_{n+1} - \hat{\tilde{x}}_n) = \phi_{\ell 0}(\hat{\tilde{x}}_n - \tilde{x}_n) + \psi_\ell(\tilde{x}_{n+1} - \hat{\tilde{x}}_n) \\ - \psi_0(\hat{\tilde{x}}_{n+1} - \hat{\tilde{x}}_n) + \psi_0(\hat{\tilde{x}}_n - \hat{\tilde{x}}_{n-1}). \end{aligned} \quad (E_{i\ell})$$

A visualization of the 2D corner equations embedded in \mathbb{Z}^3 is given in Figure 10.

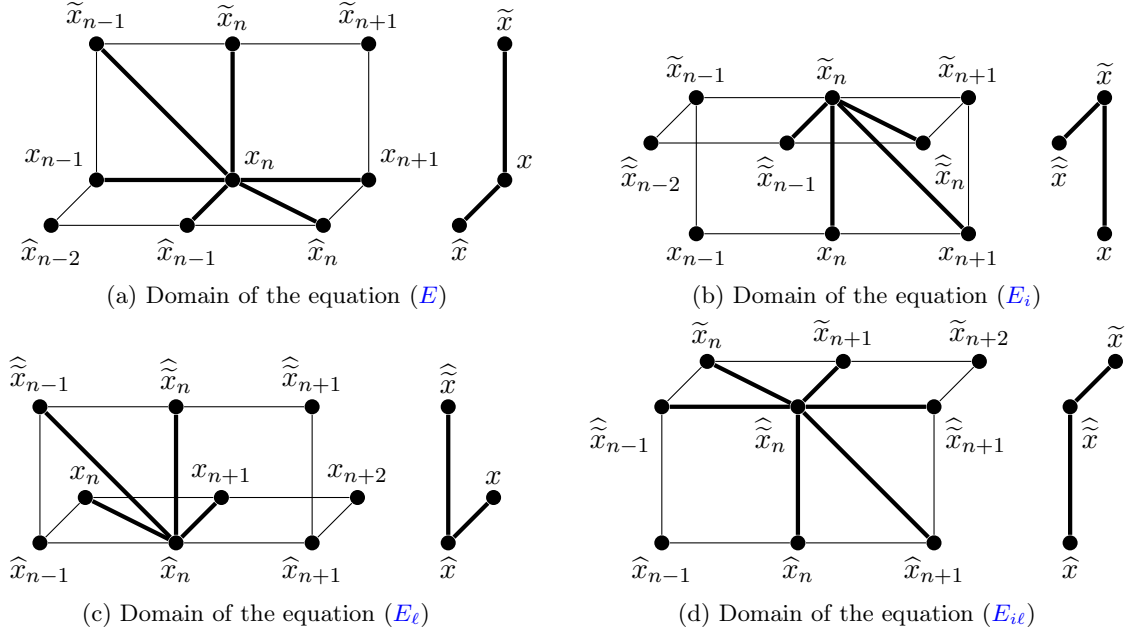


Figure 10: 2D corner equations for the maps F_i and G_ℓ : 2D corner equations (E_i), (E_ℓ) are 3D corner equations of the corresponding discrete 2-form, while each one of the 2D corner equations (E), (E_{iℓ}) is a sum of two 3D corner equations sharing one common face.

Theorem 3.6. Suppose that the fields x , \tilde{x} , and \hat{x} satisfy 2D corner equations (E). Define the fields $\hat{\tilde{x}}$ by any of the following two formulae, which are equivalent by virtue of (E):

$$\psi_i(\tilde{x}_n - x_n) + \phi_{i\ell}(x_n - \hat{\tilde{x}}_{n-1}) = \phi_{\ell 0}(\hat{x}_n - x_n) + \psi_0(x_{n+1} - x_n), \quad (S1)$$

$$\psi_0(x_{n+1} - x_n) + \phi_{i0}(x_{n+1} - \tilde{x}_n) = \psi_\ell(x_{n+1} - \hat{x}_n) + \phi_{i\ell}(x_{n+1} - \hat{\tilde{x}}_n), \quad (S2)$$

called superposition formulae. Then the 2D corner equations (E_i), (E_ℓ), (E_{iℓ}) and the following two equations are satisfied, as well:

$$\psi_0(\hat{\tilde{x}}_n - \hat{\tilde{x}}_{n-1}) + \phi_{i0}(\hat{x}_n - \hat{\tilde{x}}_{n-1}) = \psi_\ell(\tilde{x}_n - \hat{\tilde{x}}_{n-1}) + \phi_{i\ell}(x_n - \hat{\tilde{x}}_{n-1}), \quad (S3)$$

$$\psi_i(\hat{\tilde{x}}_n - \hat{x}_n) + \phi_{i\ell}(x_{n+1} - \hat{\tilde{x}}_n) = \phi_{\ell 0}(\hat{\tilde{x}}_n - \tilde{x}_n) + \psi_0(\hat{\tilde{x}}_n - \hat{\tilde{x}}_{n-1}). \quad (S4)$$

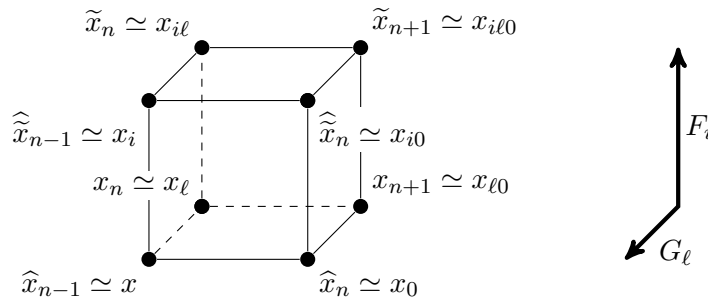


Figure 11: Identification of fields: the 0th coordinate direction enumerates the lattice cites, the coordinate directions i , ℓ correspond to the maps F_i , G_ℓ , respectively.

Proof. We see that the present theorem works in the same manner as Theorems 3.4 and 3.5, with the only difference that now we only have two local superposition formulae (S1) and (S2). We identify the fields in the way which is in Figure 11. Then equations (E_ℓ), (E_i) and (S1)–(S4) build the system of consistent 3D corner equations (\mathcal{E}_i), (\mathcal{E}_{ij}). More precisely, the correspondence is as follows:

- Equation (E_ℓ) is (\mathcal{E}_0).
- Equation (E_i) is ($\mathcal{E}_{i\ell}$).
- Equation (S1) is (\mathcal{E}_ℓ).
- Equation (S2) is ($\mathcal{E}_{0\ell}$).
- Equation (S3) is (\mathcal{E}_i).
- Equation (S4) is (\mathcal{E}_{0i}).

First, we observe that equation (E) is a sum of equation (S1) and (a downshifted version of) equation (S2). Therefore, if (E) and one of equations (S1), (S2) hold, the remaining equations of (S1), (S2) holds, too.

Due to the fact that the system of 3D corner equations has rank 2, we can claim the following: if (E) and one of equations (S1), (S2) is satisfied, then equations (E_ℓ), (E_i), (S3) and (S4) are fulfilled, as well.

Finally, we observe that equation ($E_{i\ell}$) is a sum of (an upshifted version of) equation (S3) and equation (S4). This completes the proof. \square

4 Bäcklund transformations for symmetric systems of the relativistic Toda type

We start with the pluri-Lagrangian systems related to the quad-equation Q_1^0 , Q_1^1 , and Q_3^0 from the ABS list (see [BPS14b] for further information). Each of these systems is constructed with the help of just one fundamental function $\phi(x)$, which is given for these three cases by

$$\phi(x; \alpha) = \frac{\alpha}{x}, \quad \phi(x; \alpha) = \frac{1}{2} \log \frac{x + \alpha}{x - \alpha}, \quad \text{and} \quad \phi(x; \alpha) = \frac{1}{2} \log \frac{\sinh(x + \alpha)}{\sinh(x - \alpha)}, \quad (21)$$

respectively. The corner equations are given by (\mathcal{E}_i), (\mathcal{E}_{ij}) with the leg functions

$$\psi_i(x) = \phi(x; \alpha_i), \quad \phi_{ij}(x) = \phi(x; \alpha_i - \alpha_j),$$

as well as the following choice of parameters:

$$\alpha_0 = \alpha, \quad \alpha_i = \lambda, \quad \alpha_j = \mu, \quad \alpha_k = \lambda + \alpha, \quad \alpha_\ell = \mu + \alpha.$$

Thus, the two mutually commuting families of Bäcklund transformations are given by

$$F_i : \begin{cases} p_n = \phi(\tilde{x}_n - x_n; \lambda) + \phi(x_n - \tilde{x}_{n-1}; \lambda - \alpha) - \phi(x_{n+1} - x_n; \alpha) + \phi(x_n - x_{n-1}; \alpha), \\ \tilde{p}_n = \phi(\tilde{x}_n - x_n; \lambda) + \phi(x_{n+1} - \tilde{x}_n; \lambda - \alpha), \end{cases} \quad (22)$$

and

$$G_k : \begin{cases} p_n = \phi(\tilde{x}_n - x_n; \lambda) + \phi(x_n - \tilde{x}_{n-1}; \lambda + \alpha), \\ \tilde{p}_n = \phi(\tilde{x}_n - x_n; \lambda) + \phi(x_{n+1} - \tilde{x}_n; \lambda + \alpha) - \phi(\tilde{x}_{n+1} - \tilde{x}_n; \alpha) + \phi(\tilde{x}_n - \tilde{x}_{n-1}; \alpha). \end{cases} \quad (23)$$

The functions $\phi_{ij}(x)$, $\phi_{k\ell}(x)$, and $\phi_{i\ell}(x)$ used in Theorems 3.4, 3.5, and 3.6 to prove commutativity of any two of the maps F_i , F_j , G_k , and G_ℓ , are given by

$$\phi_{ij}(x) = \phi(x; \lambda - \mu), \quad \phi_{k\ell}(x) = \phi(x; \lambda - \mu), \quad \text{and} \quad \phi_{i\ell}(x) = \phi(x; \lambda - \mu - \alpha).$$

5 Bäcklund transformations for the modified exponential system of the relativistic Toda type

The pluri-Lagrangian system playing the “master” role of the system for all the asymmetric systems of the relativistic Toda type is described in the following proposition.

Proposition 5.1. *The pluri-Lagrangian system consisting of the 3D corner equations (\mathcal{E}_i) , (\mathcal{E}_{ij}) with the leg functions*

$$\psi_j(x) = \log \frac{\alpha_j + e^x}{\beta_j}, \quad \phi_{ij}(x) = \log \frac{\beta_j - \beta_i e^x}{\alpha_j - \alpha_i e^x} \quad (24)$$

is consistent, and the corresponding discrete Lagrangian 2-form \mathcal{L} is closed on its solutions.

Proof. The system in question is a slight generalization of a similar pluri-Lagrangian system for which the analogous statements were proven in [BPS14a], and which consists of the 3D corner equations (\mathcal{E}_i) , (\mathcal{E}_{ij}) with the leg functions

$$\bar{\psi}_j(x) = \log(\gamma_j - e^x), \quad \bar{\phi}_{ij}(x) = \log \frac{\gamma_i - \gamma_j e^x}{\gamma_j - \gamma_i e^x}. \quad (25)$$

Setting

$$x_i = \bar{x}_i + \log(-\gamma_i \beta_i), \quad x_{ij} = \bar{x}_{ij} + \log(\gamma_i \gamma_j \beta_i \beta_j), \quad \text{and} \quad \alpha_i = \gamma_i^2 \beta_i, \quad (26)$$

we find:

$$\psi_j(x_{ij} - x_i) = \bar{\psi}_j(\bar{x}_{ij} - \bar{x}_i) + \log \gamma_j, \quad \phi_{ij}(x_j - x_i) = \bar{\phi}_{ij}(\bar{x}_j - \bar{x}_i) - \log(\gamma_i \gamma_j). \quad (27)$$

This yields, up to additive constants,

$$L_j(x_{ij} - x_i) = \bar{L}_j(\bar{x}_{ij} - \bar{x}_i) + \log \gamma_j(\bar{x}_{ij} - \bar{x}_i),$$

$$\Lambda_{ij}(x_j - x_i) = \bar{\Lambda}_{ij}(\bar{x}_j - \bar{x}_i) - \log(\gamma_i \gamma_j)(\bar{x}_j - \bar{x}_i).$$

Now, the closure relation for the system with the leg functions (24) is immediately seen to be equivalent to the closure relation for the system with the leg functions (25). \square

In identifying the leg functions, we will often silently perform shifts similar to (27),

$$\psi_j \rightsquigarrow \psi_j + \epsilon_j, \quad \phi_{ij} \rightsquigarrow \phi_{ij} - \epsilon_i - \epsilon_j, \quad (28)$$

which affect neither the 3D-corner equations nor the closedness of the pluri-Lagrangian 2-form \mathcal{L} .

We start with the following choice of parameters:

$$\begin{aligned} \alpha_0 &= \frac{1}{\alpha}, & \alpha_i &= -1, & \alpha_j &= -1, & \alpha_k &= \frac{1}{\lambda + \alpha}, & \alpha_\ell &= \frac{1}{\mu + \alpha}, \\ \beta_0 &= -1, & \beta_i &= \lambda, & \beta_j &= \mu, & \beta_k &= -1, & \beta_\ell &= -1. \end{aligned}$$

With this choice of parameters, the relevant leg functions for the map F_i are

$$\psi_0(x) = \log(1 + \alpha e^x), \quad \psi_i(x) = \log \frac{e^x - 1}{\lambda}, \quad \phi_{i0}(x) = \log \frac{1 + \lambda e^x}{1 + \alpha e^x},$$

and those relevant for the map G_k are

$$\psi_k(x) = \log(1 + (\lambda + \alpha)e^x), \quad \phi_{k0}(x) = \log \frac{\lambda^{-1}(e^x - 1)}{1 - \lambda^{-1}\alpha(e^x - 1)}.$$

Moreover, we always assume that the functions ψ_j, ϕ_{j0} are obtained from ψ_i, ϕ_{i0} by replacing λ by μ , and, the functions $\psi_\ell, \phi_{\ell 0}$ are obtained from ψ_k, ϕ_{k0} in the same way. Thus, we are dealing with the following two families of Bäcklund transformations:

$$F_i : \begin{cases} e^{p_n} = \frac{e^{\tilde{x}_n - x_n} - 1}{\lambda} \cdot \frac{1 + \lambda e^{x_n - \tilde{x}_{n-1}}}{1 + \alpha e^{x_n - \tilde{x}_{n-1}}} \cdot \frac{1 + \alpha e^{x_n - x_{n-1}}}{1 + \alpha e^{x_{n+1} - x_n}}, \\ e^{\tilde{p}_n} = \frac{e^{\tilde{x}_n - x_n} - 1}{\lambda} \cdot \frac{1 + \lambda e^{x_n + 1 - \tilde{x}_n}}{1 + \alpha e^{x_{n+1} - \tilde{x}_n}}; \end{cases} \quad (29)$$

and

$$G_k : \begin{cases} e^{p_n} = \frac{\lambda^{-1}(e^{\tilde{x}_n - x_n} - 1)}{1 - \lambda^{-1}\alpha(e^{\tilde{x}_n - x_n} - 1)} \cdot (1 + (\lambda + \alpha)e^{x_n - \tilde{x}_{n-1}}), \\ e^{\tilde{p}_n} = \frac{\lambda^{-1}(e^{\tilde{x}_n - x_n} - 1)}{1 - \lambda^{-1}\alpha(e^{\tilde{x}_n - x_n} - 1)} \cdot (1 + (\lambda + \alpha)e^{x_{n+1} - \tilde{x}_n}) \cdot \frac{1 + \alpha e^{\tilde{x}_n - \tilde{x}_{n-1}}}{1 + \alpha e^{\tilde{x}_{n+1} - \tilde{x}_n}}. \end{cases} \quad (30)$$

The functions $\phi_{ij}(x)$, $\phi_{k\ell}(x)$, and $\phi_{i\ell}(x)$ used in Theorems 3.4, 3.5, and 3.6 to prove commutativity of any two of the maps F_i, F_j, G_k , and G_ℓ , are given by

$$\phi_{ij}(x) = \log \frac{\lambda e^x - \mu}{e^x - 1}, \quad \phi_{k\ell}(x) = \log \frac{e^x - 1}{(\mu + \alpha)e^x - (\lambda + \alpha)}, \quad \phi_{i\ell}(x) = \log \frac{1 + \lambda e^x}{1 + (\mu + \alpha)e^x}.$$

6 Bäcklund transformations for the “master” exponential system of the relativistic Toda type

The following system is algebraically similar to the modified exponential one, but provides us with more freedom in the choice of parameters. We set in the system of Proposition 5.1:

$$\begin{aligned} \alpha_0 &= \frac{1}{\epsilon\alpha}, & \alpha_i &= \frac{\lambda - \epsilon}{\epsilon}, & \alpha_j &= \frac{\mu - \epsilon}{\epsilon}, & \alpha_k &= \frac{1}{\epsilon(\alpha + \lambda)}, & \alpha_\ell &= \frac{1}{\epsilon(\alpha + \mu)}, \\ \beta_0 &= \frac{1}{\alpha - \epsilon}, & \beta_i &= \lambda, & \beta_j &= \mu, & \beta_k &= \frac{1}{\alpha + \lambda - \epsilon}, & \beta_\ell &= \frac{1}{\alpha + \mu - \epsilon}. \end{aligned}$$

Up to the shifts of the type (28), we find:

$$\psi_0(x) = \log(1 + \epsilon\alpha e^x), \quad \psi_i(x) = \log \left(1 + \frac{\epsilon}{\lambda}(e^x - 1) \right), \quad \phi_{i0}(x) = \log \frac{1 - \lambda(\alpha - \epsilon)e^x}{1 - \alpha(\lambda - \epsilon)e^x},$$

and

$$\psi_k(x) = \log(1 + \epsilon(\lambda + \alpha)e^x), \quad \phi_{k0}(x) = \log \frac{1 - \lambda^{-1}(\alpha - \epsilon)(e^x - 1)}{1 - \lambda^{-1}\alpha(e^x - 1)},$$

so that we are dealing with the following two families of Bäcklund transformations:

$$F_i : \begin{cases} e^{\epsilon p_n} = (1 + \epsilon \lambda^{-1} (e^{\tilde{x}_n - x_n} - 1)) \cdot \frac{1 - \lambda(\alpha - \epsilon)e^{x_n - \tilde{x}_{n-1}}}{1 - \alpha(\lambda - \epsilon)e^{x_n - \tilde{x}_{n-1}}} \cdot \frac{1 + \epsilon \alpha e^{x_n - x_{n-1}}}{1 + \epsilon \alpha e^{x_{n+1} - x_n}}, \\ e^{\epsilon \tilde{p}_n} = (1 + \epsilon \lambda^{-1} (e^{\tilde{x}_n - x_n} - 1)) \cdot \frac{1 - \lambda(\alpha - \epsilon)e^{x_{n+1} - \tilde{x}_n}}{1 - \alpha(\lambda - \epsilon)e^{x_{n+1} - \tilde{x}_n}}; \end{cases} \quad (31)$$

and

$$G_k : \begin{cases} e^{\epsilon p_n} = \frac{1 - (\alpha - \epsilon)\lambda^{-1} (e^{\tilde{x}_n - x_n} - 1)}{1 - \alpha\lambda^{-1} (e^{\tilde{x}_n - x_n} - 1)} \cdot (1 + \epsilon(\alpha + \lambda)e^{x_n - \tilde{x}_{n-1}}), \\ e^{\epsilon \tilde{p}_n} = \frac{1 - (\alpha - \epsilon)\lambda^{-1} (e^{\tilde{x}_n - x_n} - 1)}{1 - \alpha\lambda^{-1} (e^{\tilde{x}_n - x_n} - 1)} \cdot \frac{1 + \epsilon \alpha e^{\tilde{x}_n - \tilde{x}_{n-1}}}{1 + \epsilon \alpha e^{\tilde{x}_{n+1} - \tilde{x}_n}} \cdot (1 + \epsilon(\alpha + \lambda)e^{x_{n+1} - \tilde{x}_n}). \end{cases} \quad (32)$$

The functions $\phi_{ij}(x)$, $\phi_{k\ell}(x)$, and $\phi_{i\ell}(x)$ used in Theorems 3.4, 3.5, and 3.6 to prove commutativity of any two of the maps F_i , F_j , G_k , and G_ℓ , are given by

$$\begin{aligned} \phi_{ij}(x) &= \log \frac{\lambda e^x - \mu}{(\lambda - \epsilon)e^x - (\mu - \epsilon)}, & \phi_{k\ell}(x) &= \log \frac{(\mu + \alpha - \epsilon)e^x - (\lambda + \alpha - \epsilon)}{(\mu + \alpha)e^x - (\lambda + \alpha)}, \\ \phi_{i\ell}(x) &= \log \frac{1 - \lambda(\mu + \alpha - \epsilon)e^x}{1 - (\lambda - \epsilon)(\mu + \alpha)e^x}. \end{aligned}$$

7 Bäcklund transformations for the additive exponential system of the relativistic Toda type

Performing the limit $\epsilon \rightarrow 0$ in the pluri-Lagrangian system of Section 6, we arrive at the following leg functions:

$$\psi_0(x) = \alpha e^x, \quad \psi_i(x) = \frac{1}{\lambda}(e^x - 1), \quad \phi_{i0}(x) = \frac{(\lambda - \alpha)e^x}{1 - \lambda \alpha e^x},$$

and

$$\psi_k(x) = (\lambda + \alpha)e^x, \quad \phi_{k0}(x) = \frac{\lambda^{-1}(e^x - 1)}{1 - \lambda^{-1}\alpha(e^x - 1)}.$$

Thus, we are dealing with the following two families of Bäcklund transformations:

$$F_i : \begin{cases} p_n = \frac{e^{\tilde{x}_n - x_n} - 1}{\lambda} + \frac{(\lambda - \alpha)e^{x_n - \tilde{x}_{n-1}}}{1 - \lambda \alpha e^{x_n - \tilde{x}_{n-1}}} + \alpha e^{x_n - x_{n-1}} - \alpha e^{x_{n+1} - x_n}, \\ \tilde{p}_n = \frac{e^{\tilde{x}_n - x_n} - 1}{\lambda} + \frac{(\lambda - \alpha)e^{x_{n+1} - \tilde{x}_n}}{1 - \lambda \alpha e^{x_{n+1} - \tilde{x}_n}}; \end{cases} \quad (33)$$

and

$$G_k : \begin{cases} p_n = \frac{\lambda^{-1} (e^{\tilde{x}_n - x_n} - 1)}{1 - \alpha \lambda^{-1} (e^{\tilde{x}_n - x_n} - 1)} + (\lambda + \alpha)e^{x_n - \tilde{x}_{n-1}}, \\ \tilde{p}_n = \frac{\lambda^{-1} (e^{\tilde{x}_n - x_n} - 1)}{1 - \alpha \lambda^{-1} (e^{\tilde{x}_n - x_n} - 1)} + (\lambda + \alpha)e^{x_{n+1} - \tilde{x}_n} - \alpha e^{\tilde{x}_{n+1} - \tilde{x}_n} + \alpha e^{\tilde{x}_n - \tilde{x}_{n-1}}. \end{cases} \quad (34)$$

Any two of the symplectic maps F_i , F_j , G_k , and G_ℓ commute, which is demonstrated in Theorems 3.4, 3.5, and 3.6, with the following functions ϕ_{ij} , $\phi_{k\ell}$, and $\phi_{i\ell}$:

$$\phi_{ij}(x) = \frac{e^x - 1}{\lambda e^x - \mu}, \quad \phi_{k\ell}(x) = -\frac{e^x - 1}{(\mu + \alpha)e^x - (\lambda + \alpha)}, \quad \phi_{i\ell}(x) = \frac{(\lambda - \mu - \alpha)e^x}{1 - \lambda(\mu + \alpha)e^x}.$$

8 Bäcklund transformations for the asymmetric rational system of the relativistic Toda type

The last pluri-Lagrangian system we consider in the present paper is described in the following proposition.

Proposition 8.1. *The system of 3D-corner equations (\mathcal{E}_i) , (\mathcal{E}_{ij}) with the leg functions*

$$\psi_j(x) = \log(x + \alpha_j), \quad \phi_{ij}(x) = \log \frac{x + \beta_j - \beta_i}{x + \alpha_j - \alpha_i} \quad (35)$$

is consistent, and the corresponding discrete Lagrangian 2-form \mathcal{L} is closed on its solutions.

Proof. The leg functions (35) are obtained from those in (24) via the following changes of variables and transformations of parameters:

$$x \rightsquigarrow hx, \quad \alpha_i \rightsquigarrow -1 + h\alpha_i, \quad \beta_i \rightsquigarrow -1 + h\beta_i,$$

and then sending $h \rightarrow 0$. □

We make the following choice of parameters:

$$\begin{aligned} \alpha_0 &= \frac{1}{\alpha}, & \alpha_i &= 0, & \alpha_j &= 0, & \alpha_k &= \frac{1}{\alpha + \lambda}, & \alpha_\ell &= \frac{1}{\alpha + \mu}, \\ \beta_0 &= 0, & \beta_i &= -\frac{1}{\lambda}, & \beta_j &= -\frac{1}{\mu}, & \beta_k &= 0, & \beta_\ell &= 0. \end{aligned}$$

With this choice of parameters, we find the following leg functions:

$$\psi_0(x) = \log(1 + \alpha x), \quad \psi_i(x) = \log \frac{x}{\lambda}, \quad \phi_{i0}(x) = \log \frac{1 + \lambda x}{1 + \alpha x},$$

and

$$\psi_k(x) = \log(1 + (\lambda + \alpha)x), \quad \phi_{k0}(x) = \log \frac{x}{\lambda + \alpha(\lambda + \alpha)x}.$$

This corresponds to the following two families of Bäcklund transformations:

$$F_i : \begin{cases} e^{p_n} = \frac{\tilde{x}_n - x_n}{\lambda} \cdot \frac{1 + \lambda(x_n - \tilde{x}_{n-1})}{1 + \alpha(x_n - \tilde{x}_{n-1})} \cdot \frac{1 + \alpha(x_n - x_{n-1})}{1 + \alpha(x_{n+1} - x_n)}, \\ e^{\tilde{p}_n} = \frac{\tilde{x}_n - x_n}{\lambda} \cdot \frac{1 + \lambda(x_{n+1} - \tilde{x}_n)}{1 + \alpha(x_{n+1} - \tilde{x}_n)}. \end{cases} \quad (36)$$

and

$$G_k : \begin{cases} e^{p_n} = \frac{\tilde{x}_n - x_n}{\lambda + \alpha(\lambda + \alpha)(\tilde{x}_n - x_n)} \cdot (1 + (\lambda + \alpha)(x_n - \tilde{x}_{n-1})), \\ e^{\tilde{p}_n} = \frac{\tilde{x}_n - x_n}{\lambda + \alpha(\lambda + \alpha)(\tilde{x}_n - x_n)} \cdot (1 + (\lambda + \alpha)(x_{n+1} - \tilde{x}_n)) \cdot \frac{1 + \alpha(\tilde{x}_n - \tilde{x}_{n-1})}{1 + \alpha(\tilde{x}_{n+1} - \tilde{x}_n)}. \end{cases} \quad (37)$$

Any two of the symplectic maps F_i , F_j , G_k , and G_ℓ commute, as follows from Theorems 3.4, 3.5, and 3.6 with the functions

$$\begin{aligned} \phi_{ij}(x) &= \log \frac{\mu - \lambda + \lambda\mu x}{x}, & \phi_{k\ell}(x) &= \log \frac{x}{\lambda - \mu + (\lambda + \alpha)(\mu + \alpha)x}, \\ \phi_{i\ell}(x) &= \log \frac{1 + \lambda x}{1 + (\mu + \alpha)x}. \end{aligned}$$

9 Spectrality, integrals of motion, and conservation laws

In all our examples the maps F_i and G_k as introduced in Definitions 3.1 and 3.2 are instances of *Bäcklund transformations*, i.e., one-parameter families of commuting symplectic maps characterized by parameter-dependent Lagrange functions $\mathfrak{L}(x, \tilde{x}; \lambda)$, resp. $\mathfrak{M}(x, \tilde{x}; \lambda)$. For such families, the following remarkable property was established in [Sur03]:

Theorem 9.1. *For the pluri-Lagrangian system built by two commuting maps F_i, F_j from one family of Bäcklund transformations (whose action is denoted by \sim and $\hat{\cdot}$, respectively), the discrete multi-time Lagrangian 1-form is closed on solutions if and only if the quantity $P_F(x, \tilde{x}; \lambda) := \partial \mathfrak{L}(x, \tilde{x}; \lambda) / \partial \lambda$ is a common integral of motion for all F_j .*

This is a re-formulation of the mysterious “spectrality property” of Bäcklund transformations discovered by Kuznetsov and Sklyanin [KS98]. In the specific context of relativistic Toda-type systems, we are actually dealing with two mutually commuting families of Bäcklund transformations. In this situation, the following weaker statement can be made.

Theorem 9.2. *For the pluri-Lagrangian system built by two commuting maps G_k and F_j (whose action is denoted by \sim and $\hat{\cdot}$, respectively) from two mutually commuting families of Bäcklund transformations, if the discrete multi-time Lagrangian 1-form is closed on solutions, then the quantity $P_G(x, \tilde{x}; \lambda) := \partial \mathfrak{M}(x, \tilde{x}; \lambda) / \partial \lambda$ is a common integral of motion for all F_j .*

Proof. The closedness of the discrete multi-time Lagrangian 1-form is expressed by the following formula:

$$\mathfrak{M}(x, \tilde{x}; \lambda) + \mathfrak{L}(\tilde{x}, \hat{\tilde{x}}; \mu) - \mathfrak{M}(\hat{\tilde{x}}, \hat{\tilde{x}}; \lambda) - \mathfrak{L}(x, \hat{x}; \mu) = 0.$$

Differentiating with respect to λ and taking into account that the terms with $\partial \tilde{x} / \partial \lambda$ etc. vanish due to the corresponding corner equations, we arrive at

$$\frac{\partial \mathfrak{M}(\hat{\tilde{x}}, \hat{\tilde{x}}; \lambda)}{\partial \lambda} - \frac{\partial \mathfrak{M}(x, \tilde{x}; \lambda)}{\partial \lambda} = 0,$$

which is the required property. \square

Thus, for the maps F_i and G_k as introduced in Definitions 3.1 and 3.2, with the understanding that the corresponding Lagrangian depend on the parameters, as in the examples discussed above, i.e.,

$$\mathfrak{L}(x, \tilde{x}; \lambda) = \sum_n L_i(\tilde{x}_n - x_n; \lambda) - \sum_n L_0(x_{n+1} - x_n; \alpha) - \sum_n \Lambda_{i0}(x_{n+1} - \tilde{x}_n; \lambda, \alpha), \quad (38)$$

$$\mathfrak{M}(x, \tilde{x}; \lambda) = \sum_n \Lambda_{k0}(\tilde{x}_n - x_n; \lambda, \alpha) + \sum_n L_0(\tilde{x}_n - \tilde{x}_{n-1}; \alpha) - \sum_n L_k(x_n - \tilde{x}_{n-1}; \lambda), \quad (39)$$

we arrive at the following generating functions of common integrals of all the maps F_j, G_ℓ :

$$P_F(x, \tilde{x}; \lambda) = \sum_n \frac{\partial L_i(\tilde{x}_n - x_n; \lambda)}{\partial \lambda} - \sum_n \frac{\partial \Lambda_{i0}(x_{n+1} - \tilde{x}_n; \lambda, \alpha)}{\partial \lambda}, \quad (40)$$

$$P_G(x, \tilde{x}; \lambda) = \sum_n \frac{\partial \Lambda_{k0}(\tilde{x}_n - x_n; \lambda, \alpha)}{\partial \lambda} - \sum_n \frac{\partial L_k(x_n - \tilde{x}_{n-1}; \lambda)}{\partial \lambda}. \quad (41)$$

We are going to demonstrate that the 2-dimensional pluri-Lagrangian interpretation allows us to get additional important information. Namely, we can derive the *local form* of the fact that P_F and P_G are the integrals of motion. For the sake of simplicity, we restrict ourselves to the local form of the statement that $P_F(x, \tilde{x}; \lambda)$ is an integral of motion of the maps F_j .

Theorem 9.3. *If the three-point discrete 2-form \mathcal{L} is closed on solutions of the system of 3D corner equations corresponding to the maps F_i and F_j , then the latter system admits the conservation law*

$$\Delta_j P_{i0} = \Delta_0 P_{ij}, \quad (42)$$

with the densities

$$P_{i0} = \frac{\partial L_i(\tilde{x}_n - x_n; \lambda)}{\partial \lambda} - \frac{\partial \Lambda_{i0}(x_{n+1} - \tilde{x}_n; \lambda, \alpha)}{\partial \lambda}, \quad (43)$$

$$P_{ij} = \frac{\partial L_i(\tilde{x}_n - x_n; \lambda)}{\partial \lambda} - \frac{\partial \Lambda_{ij}(\hat{x}_n - \tilde{x}_n; \lambda, \mu)}{\partial \lambda}. \quad (44)$$

Observe that P_{i0} is the summand in (40).

(Recall that $\Delta_j = T_j - I$, $\Delta_0 = T_0 - I$, where T_j is the shift corresponding to the map F_j and denoted by $\hat{\cdot}$, while T_0 is the shift $n \rightarrow n+1$.)

Proof. We start by re-writing expression (13) for $d\mathcal{L}$ in the form specific for our present context:

$$\begin{aligned} d\mathcal{L} = S^{ij0} = & L_i(\tilde{x}_{n+1} - x_{n+1}; \lambda) + L_j(\hat{x}_n - \tilde{x}_n; \mu) + L_0(\hat{x}_{n+1} - \hat{x}_n; \alpha) \\ & - L_i(\hat{x}_n - \hat{x}_n; \lambda) - L_j(\hat{x}_{n+1} - x_{n+1}; \mu) - L_0(\tilde{x}_{n+1} - \tilde{x}_n; \alpha) \\ & - \Lambda_{ij}(\hat{x}_{n+1} - \tilde{x}_{n+1}; \lambda, \mu) - \Lambda_{j0}(\tilde{x}_{n+1} - \hat{x}_n; \mu, \alpha) + \Lambda_{i0}(\hat{x}_{n+1} - \hat{x}_n; \lambda, \alpha) \\ & + \Lambda_{ij}(\hat{x}_n - \tilde{x}_n; \lambda, \mu) + \Lambda_{j0}(x_{n+1} - \hat{x}_n; \mu, \alpha) - \Lambda_{i0}(x_{n+1} - \tilde{x}_n; \lambda, \alpha) = 0. \end{aligned} \quad (45)$$

Differentiating equation (45) with respect to λ and taking into account that the terms containing $\partial \tilde{x}_n / \partial \lambda$ etc. vanish by virtue of the corresponding 3D corner equations, we arrive at

$$\begin{aligned} & \frac{\partial L_i(\tilde{x}_{n+1} - x_{n+1}; \lambda)}{\partial \lambda} - \frac{\partial L_i(\hat{x}_n - \hat{x}_n; \lambda)}{\partial \lambda} \\ & - \frac{\partial \Lambda_{ij}(\hat{x}_{n+1} - \tilde{x}_{n+1}; \lambda, \mu)}{\partial \lambda} + \frac{\partial \Lambda_{i0}(\hat{x}_{n+1} - \hat{x}_n; \lambda, \alpha)}{\partial \lambda} \\ & + \frac{\partial \Lambda_{ij}(\hat{x}_n - \tilde{x}_n; \lambda, \mu)}{\partial \lambda} - \frac{\partial \Lambda_{i0}(x_{n+1} - \tilde{x}_n; \lambda, \alpha)}{\partial \lambda} = 0. \end{aligned}$$

This is equivalent to the statement of the theorem. \square

Example: Modified exponential system. We start with the maps F_i . An easy computation shows:

$$\frac{\partial L_i(x; \lambda)}{\partial \lambda} = -\frac{x}{\lambda}, \quad \frac{\partial \Lambda_{i0}(x; \lambda, \alpha)}{\partial \lambda} = \frac{1}{\lambda} \log(1 + \lambda e^x).$$

Thus, Theorems 9.1 and 9.2 lead to the following generating function of integrals of motion for all maps F_j , G_ℓ :

$$\frac{\partial \mathfrak{L}(x, \tilde{x}; \lambda)}{\partial \lambda} = \log P_F(x, \tilde{x}; \lambda),$$

where

$$P_F(x, \tilde{x}; \lambda) = \prod_{n=1}^N e^{\tilde{x}_n - x_n} (1 + \lambda e^{x_{n+1} - \tilde{x}_n}). \quad (46)$$

It is instructive to have a look at the local form of this result provided by Theorem 9.3. We compute:

$$\frac{\partial \Lambda_{ij}(x; \lambda, \mu)}{\partial \lambda} = \frac{1}{\lambda} \log(\lambda e^x - \mu),$$

and end up with the conservation law (42) with the densities

$$P_{i0} = \log e^{\tilde{x}_n - x_n} \left(1 + \lambda e^{x_{n+1} - \tilde{x}_n}\right), \quad P_{ij} = \log e^{\tilde{x}_n - x_n} \left(\lambda e^{\tilde{x}_n - \tilde{x}_n} - \mu\right). \quad (47)$$

We can give a nice expressions for the generating function of integrals $P_F(x, \tilde{x}; \lambda)$ in terms of the canonically conjugate variables x, p .

Theorem 9.4. *Set*

$$U_n(x, p; \lambda) = \begin{pmatrix} 1 + \lambda(e^{p_n} + e^{x_n - x_{n-1}}) & -\lambda(\lambda - \alpha)e^{x_n - x_{n-1} + p_{n-1}} \\ 1 & 0 \end{pmatrix}, \quad (48)$$

and further

$$T_N(x, p; \lambda) = U_N(x, p; \lambda) \dots U_2(x, p; \lambda) U_1(x, p; \lambda).$$

Then, in the periodic case $P_F(x, \tilde{x}; \lambda)$ is an eigenvalue of the matrix $T_N(x, p; \lambda)$, while in the open-end case $P_F(x, \tilde{x}; \lambda)$ is the $(1,1)$ -entry of the matrix $T_N(x, p; \lambda)$.

Proof. Set

$$\gamma_n = e^{\tilde{x}_n - x_n} \left(1 + \lambda e^{x_n - \tilde{x}_{n-1}}\right) \frac{1 + \alpha e^{x_{n+1} - \tilde{x}_n}}{1 + \alpha e^{x_{n+1} - x_n}} \cdot \frac{1 + \alpha e^{x_n - x_{n-1}}}{1 + \alpha e^{x_n - \tilde{x}_{n-1}}},$$

so that $\gamma_N \dots \gamma_2 \gamma_1 = P_F(x, \tilde{x}; \lambda)$ in both the periodic and the open-end cases. By a straightforward computation based on the first formula in (29) one checks that

$$U_n(x, p; \lambda)[\gamma_{n-1}] = \gamma_n,$$

where we write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} [z] := \frac{az + b}{cz + d}.$$

Equivalently,

$$U_n(x, p; \lambda) \begin{pmatrix} \gamma_{n-1} \\ 1 \end{pmatrix} \sim \begin{pmatrix} \gamma_n \\ 1 \end{pmatrix}.$$

The proportionality coefficient can be determined by comparing the second components of these vectors, which results in

$$U_n(x, p; \lambda) \begin{pmatrix} \gamma_{n-1} \\ 1 \end{pmatrix} = \gamma_{n-1} \begin{pmatrix} \gamma_n \\ 1 \end{pmatrix}. \quad (49)$$

Thus, in the periodic case $\gamma_N \dots \gamma_2 \gamma_1$ is the eigenvalue of $T_N(x, p; \lambda)$ corresponding to the eigenvector $\begin{pmatrix} \gamma_0 \\ 1 \end{pmatrix}$. In the open end case, equation (49) holds true for $2 \leq n \leq N$, and has to be supplemented with

$$U_1(x, p; \lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \gamma_N \\ 1 \end{pmatrix} = \gamma_N.$$

This yields

$$\begin{pmatrix} 1 & 0 \end{pmatrix} T_N(x, p; \lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \gamma_N \cdots \gamma_2 \gamma_1.$$

□

Turning now to the maps G_k , we find:

$$\frac{\partial \Lambda_{k0}(x; \lambda, \alpha)}{\partial \lambda} = -\frac{1}{\lambda + \alpha} x + \frac{1}{\lambda + \alpha} \log \left(1 - \frac{\alpha}{\lambda} (e^x - 1) \right), \quad (50)$$

$$\frac{\partial L_k(x; \lambda)}{\partial \lambda} = \frac{1}{\lambda + \alpha} \log (1 + (\lambda + \alpha) e^x). \quad (51)$$

According to Theorems 9.1 and 9.2, we compute:

$$\frac{\partial \mathfrak{M}(x, \tilde{x}; \lambda)}{\partial \lambda} = -\frac{1}{\lambda + \alpha} \log P_G(x, \tilde{x}; \lambda),$$

where

$$P_G(x, \tilde{x}; \lambda) = \prod_{n=1}^N \frac{e^{\tilde{x}_n - x_n}}{1 - \frac{\alpha}{\lambda} (e^{\tilde{x}_n - x_n} - 1)} \left(1 + (\lambda + \alpha) e^{x_n - \tilde{x}_{n-1}} \right). \quad (52)$$

Again, $P_G(x, \tilde{x}; \lambda)$ is a generating function of common integrals of motion for all the maps F_j , G_ℓ . Remarkably, in terms of the canonically conjugate variables x, p this is essentially the same function as $P_F(x, \tilde{x}; \lambda)$.

Theorem 9.5. *In the periodic case, $P_G(x, \tilde{x}; \lambda)$ is an eigenvalue of the matrix $T_N(x, p; \lambda + \alpha)$, while in the open-end case $P_G(x, \tilde{x}; \lambda)$ is the $(1, 1)$ -entry of the matrix $T_N(x, p; \lambda + \alpha)$.*

Proof. Set

$$\beta_n = \frac{e^{\tilde{x}_n - x_n}}{1 - \frac{\alpha}{\lambda} (e^{\tilde{x}_n - x_n} - 1)} \left(1 + (\lambda + \alpha) e^{x_n - \tilde{x}_{n-1}} \right).$$

Then one easily checks with the help of the first formula in (30) that

$$U_n(x, p; \lambda + \alpha) [\beta_{n-1}] = \beta_n.$$

Then, the proof goes along the same lines as the proof of Theorem 9.4. □

10 Conclusions

In the present paper, we applied the general theory of two-dimensional pluri-Lagrangian systems, developed in [BPS14a], to the analysis of Bäcklund transformations for relativistic Toda-type systems. It was possible due to a novel way to embed the one-dimensional relativistic Toda-type systems into certain two-dimensional lattice systems. This embedding is well suited for a proof of commutativity of Bäcklund transformations, as well as for the proof of the closure relation of the corresponding action functional. A different relation of this kind between relativistic Toda-type systems and 3D consistent systems of quad-equations was discussed previously, cf. [BS02, AS04, BS10b], and led to a systematic derivation of zero-curvature representations for (discrete and continuous) relativistic and non-relativistic Toda-type systems. A connection between these two approaches is presently unclear and remains to be worked out in detail.

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